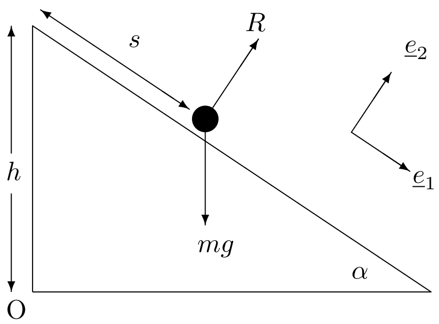
MATH1302 Example Sheet 1.

Questions 1, 2 and 4 due in Friday 28.1 *before* the lecture.

Q1 A ball mass m slides down a smooth (fixed) plane inclined at an angle α to the horizontal. The height of the plane from the lowest point is h. Take unit vectors $\underline{e}_1$ parallel to the plane and $\underline{e}_2$ perpendicular to the plane, and $s(t)$ the distance of the ball down the slope at time t. Assume the ball starts from the top of the slope at rest at time $t = 0$.



(i) Write down the acceleration $\underline{a}$ of the ball

(ii) Write down the total force $\underline{F}$ acting on it

(iii) Hence show that $\ddot{s} = g \sin\alpha$.

(iv) What is the magnitude of the reaction force on the ball from the plane?

(v) Find the distance travelled down the slope in time t

(vi) How long does the ball take to reach the bottom of the slope?

Q2 A car of mass m brakes to stop by a combination of Coulomb's and Stokes' frictional damping, so that the total resistance is $-\mu mg - k\dot{x}$ ($\dot{x} > 0$), where x is the distance from the point at which the brakes are applied, and $\mu > 0$, $k \geq 0$ constants. Show that the car braking at speed v_0 will come to rest at a time

$$T = \frac{m}{k} \log\left(1 + \frac{k}{m} t_{\text{stop}}\right),$$

where t_{stop} is the stopping time for pure Coulomb damping ($k = 0$) from the same initial speed v_0.

Q3 Sketch the curve $y(x) = \log(1 + \cos x)$ on the interval $(0, \pi)$. Taking the signed arclength $s = 0$ at $x = 0$, show that

$$x(s) = 4\tan^{-1}\left(\tanh\frac{s}{4}\right),$$

and find an expression for $y(s)$. Find also ψ as a function of s.

Q4 Show that the radius of curvature ρ of a plane curve defined by $y(x)$ can be written as

$$\rho = \frac{\left(1 + (\mathrm{d}y/\mathrm{d}x)^2\right)^{(3/2)}}{|\mathrm{d}^2y/\mathrm{d}x^2|}.$$

[Hint: write $\mathrm{d}s/\mathrm{d}\psi = (\mathrm{d}s/\mathrm{d}x)(\mathrm{d}x/\mathrm{d}\psi)$ and find $\mathrm{d}\psi/\mathrm{d}x$ from differentiating $\tan\psi = \mathrm{d}y/\mathrm{d}x$.]

Show also that for a curve given parametrically by $x = x(t)$, $y = y(t)$,

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{(3/2)}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}.$$

Hence show that the radius of curvature of an ellipse with semi-major axis a lying on the x-axis and semi-minor axis b lying on the y-axis is

$$\rho = \frac{1}{ab}\left(b^2 + \left(\frac{a^2}{b^2} - 1\right)y^2\right)^{(3/2)}.$$

Q5 A curve C is given parametrically by $x = \theta + \sin\theta$, $y = 1 - \cos\theta$ on the interval $\theta \in (-\pi, \pi)$. Show that if s is the signed arclength from the origin and ψ the angle between the tangent and the positive $x-$axis then $s = 4\sin\psi$.

A smooth bowl is made from the surface of revolution formed by rotating the above curve C about the vertical axis through the origin. A heavy particle initially sits at the bottom of the bowl. It is struck so that its initial speed is U. Find the period of the ensuing oscillation.

MATH1302 Example Sheet 2.

Questions 2, 4 and 7 due in to 802A *before* the lecture on Friday 4th F

Q1 A particle of unit mass moves in 3 dimensional space so that its position vector at time t is $\underline{r}(t) = a\cos\omega t\,\underline{i} + b\sin\omega t\,\underline{j} + ct\underline{k}$ where $a, b, c > 0$ and $a > b$. Find (i) the particle's velocity, (ii) its acceleration, (iii) sketch its speed as a function of t and finally (iv) sketch also its path.

Q2 A heavy bead moves on a smooth and strictly convex curve $y = y(x)$ for $x \in \mathrm{R}$. The curve $y(x)$ also satisfies $y(x) \to \infty$ as $|x| \to \infty$ and has its (unique) minimum at $x = 0$. Show that if ρ_0 is the radius of curvature at $x = 0$ then the period of small oscillations about the minimum is approximately $2\pi/\omega$ where $\omega^2 = g/\rho_0$.

[Note: A curve $y = y(x)$ is strictly convex if $y''(x) > 0$ for all $x \in \mathrm{R}$.]

[Hint: do a Maclaurin expansion of $\psi(s)$.]

Q3 A heavy particle B rests at the lowest point A of the inside of a fixed smooth spherical shell centre O and radius a. The particle is then struck so that its initial speed is U. Find the reaction force as a function of the angle θ that the line OA makes with OB. What are the conditions on U that the particle never leaves the inner surface of the sphere?

Q4 A particle moves at a constant speed u on a plane curve γ, and the particle's component of acceleration along the $y-$axis is constant. Show that γ is a cycloid.

Q5 An aeroplane flying in a straight line at constant height H and speed U needs to drop supplies on a target at ground level. Given that the aeroplane eventually flies directly over the target, what distance before the target does it need to release the supply package if it is to hit the target? (Neglect air resistance.)

Q6 A heavy particle is fired with speed U from the base of an inclined plane. The plane is inclined at an angle α to the horizontal. Find the distance up the plane that the particle lands as a function of the angle θ between the inclined surface of the plane and the initial direction of the particle. (Neglect air resistance.)

Q7 A heavy particle is projected at speed U at an angle α to the horizontal. The particle is subject to air resistance which is experimentally found to vary proportionally to the square of the speed. Show that

$$\dot{\underline{v}} = -\frac{g}{V^2}|\underline{v}|\underline{v} - g\underline{j},$$

where V is the terminal velocity of the particle. If $\alpha = \frac{\pi}{2}$ (so that the particle is projected directly upwards), find the maximum height reached and the time taken to reach it. What is the speed of the particle when it returns to the horizontal?

Q8 Consider the three-dimensional curve defined as

$$\underline{r}(s) = \alpha\cos s\underline{i} + \alpha\sin s\underline{j} + \gamma s\underline{k}$$

with $\alpha > 0$, $\gamma > 0$ and $\alpha^2 + \gamma^2 = 1$.

(i) Find the unit tangent vector $\underline{e}_t$ and verify that s is the arclength.

(ii) Find the normal vector $\underline{e}_n$ and give the curvature κ.

(iii) Calculate the binormal vector $\underline{e}_b$.

(iv) Verify the second and third Serret-Frenet formulae for this curve, and give the torsion τ.

MATH1302 Example Sheet 3.

Questions 1, 3 and 4 due in to 802A *before* the lecture on Friday 11 Fe

Q1 A particle P of mass m moves in the x-y plane under the action of an attractive central force $\underline{F}$. Show from the plane polars definition $h = r^2\dot{\theta}$ that its angular momentum per unit mass is $x\dot{y} - y\dot{x}$. Write down expressions in Cartesian coordinates for the accelerations $\ddot{x}$ and $\ddot{y}$. Hence show that the particle's angular momentum is conserved.

Q2 (The orbits of planets about the sun are planar)
Consider the three dimensional motion of a planet about the sun. Ignore the gravitational effects of other planets. Take the sun as the origin and let $\underline{r} = (x(t), y(t), z(t))$ be the position vector of a planet at time t. Show that the angular momentum $\underline{L} = m\underline{r} \wedge \dot{\underline{r}}$ of the planet about the sun is constant. Show also that the planet moves in a plane containing the sun. [Hint: use $\underline{L}$.]

Q3 State the formulae for radial and transverse acceleration of a particle in polar coordinates r, θ. For the motion of a particle position vector $\underline{r}$ under an attractive central force of magnitude $F(r, \theta)$ derive the equation for $u = 1/r$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{F(1/u, \theta)}{mh^2u^2}.$$

What is the constant h?

A particle P of unit mass moves with position vector $\underline{r}$ under a central force

$$\underline{F} = -\frac{k(2 + \sin 2\theta)^{-\frac{3}{2}}}{r^3}\underline{r}.$$

Show that the particle moves in orbits whose Cartesian equation are

$$1 - Ax - By = \frac{k\sqrt{2}}{3h^2}(x^2 + xy + y^2)^{1/2},$$

where h, A and B are constants. (Hint: In the $u-$equation for motion look for a particular integral proportional to $\sqrt{2 + \sin 2\theta}$ and then convert into Cartesian coordinates.)

Q4 A particle with position vector $\underline{r}$ moves under a force

$$\underline{F} = \frac{k}{r^4}\underline{r}$$

per unit mass, where $r = |\underline{r}|$ and k is *any* real number. The particle starts at $r = a$ with radial velocity $U > 0$ and tangential velocity $V > 0$. Find the equation for the particle's path in polar coordinates and discuss whether the particle motion is bounded or unbounded, being careful to consider the outcomes for different k.

Q5 A meteorite is approaching the Earth and is first detected at a very large distance away (i.e. $r = \infty$) when it is moving with speed U. If the meteorite were to continue in the absence of the Earth's gravitational pull it would pass the Earth at a minimum distance d. If G is the gravitational constant and M the mass of the Earth, find the nearest that the meteorite comes to the Earth when subject to its gravitational pull.

MATH1302 Example Sheet 4.

Questions 1, 2, 4 and 5 due in at the start of the lecture on Friday 25 F

Q1 A surface is given parametrically in Cartesian coordinates by

$$\underline{r} = ((\phi + \sin\phi)\cos\theta, (\phi + \sin\phi)\sin\theta, (1 - \cos\phi)), \quad \phi \in (-\pi, \pi)\,, \theta \in [0, 2\pi).$$

Find the equations for the surface in cylindrical coordinates ρ, θ, z (in terms of the parameter ϕ) and sketch the surface.

Q2 A heavy bead slides down a frictionless helical wire whose shape is given by $x = a\cos\theta$, $y = a\sin\theta$, $z = -b\,\theta$ for a, $b > 0$. If the bead starts from rest at $z = 0$, find its speed when it has rotated about the $z-$axis n times. [Hint: use energy conservation.]

Q3 (Conical pendulum) A pendulum consists of a light string of length a fixed at O and with a mass m attached to the free end A. The string is held so that OA makes an angle α to the downward vertical and then set in motion so that the mass m undergoes horizontal circular motion. What is the period of this circular motion?

Q4 A heavy particle moves inside a frictionless hollow sphere of radius a. Show that its motion satisfies

$$\frac{1}{2}m\left(\frac{a^2\dot{z}^2}{a^2 - z^2} + \frac{h^2}{a^2 - z^2}\right) + mgz = \text{constant},$$

where z is measured upwards from the centre of the sphere, and h is a constant. The particle is held on the inner surface of the sphere at $z = -a/2$ and given a horizontal velocity U. If $U^2 = 4ag$, find the maximum and minimum height that the particle reaches during the ensuing motion.

Q5 (2006 exam, question 4).

A particle of unit mass moves on the outside surface $z = K(r)$ of a smooth axisymmetric body, where r measures distance from the axis of symmetry (which is vertical) and z is distance measured upwards along the axis. Using conservation of energy and angular momentum, or otherwise, derive the governing equations

$$\frac{1}{2}\left\{[1 + K'(r)^2]\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{h^2}{r^2}\right\} + gK(r) = \text{constant},$$

$$r^2\dot{\theta} = h = \text{constant},$$

with t, θ denoting time and azimuthal angle respectively.

Applying the equations to a cone of semi-vertical angle α ($< \pi/2$), with vertex uppermost, show that

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \left(\frac{h^2}{r^3} + g\cot\alpha\right)\sin^2\alpha.$$

Deduce that the reaction R between the particle and the cone is given by

$$R = (g\tan\alpha - h^2/r^3)\cos\alpha.$$

MATH1302 Example Sheet 5.

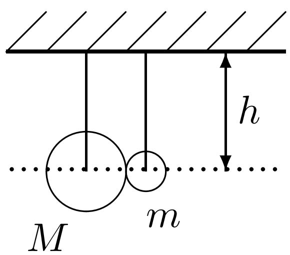
Questions 2, 5 and 6 due in Friday 4 March at 11am in the lecture or to 802

Q1 Two particles of mass m and m' move along the $x-$axis under their mutual attractive force F. The particles start at $x = 0$ with velocities u and u' respectively with $u' > u$. Show that their centre of mass moves with constant velocity and find that velocity. If the force F is constant, show that they will collide in finite time and find that time.

Q2 A marble dropped vertically onto smooth ground rebounds vertically upwards with speed U. If $e \in (0, 1)$ is the coefficient of restitution, show that the time taken from when it was dropped to the nth rebound is $\dfrac{U}{g}\dfrac{(1 + e - 2e^n)}{e(1 - e)}$. How long does it take before the ball has effectively come to a rest?

Q3 Three snooker balls A, B, C each of mass m lie in order on a straight line. Balls B and C are initially at rest and A has speed U directly towards B. If the coefficient of restitution at each impact is $1/5$ show that there will be just three impacts and find the final speed of each ball.

Q4 Consider the version of Newton's cradle shown:



Two masses, m and M hang vertically so that their centres of mass are vertically below their supports and so that they are just touching. The mass m is pulled to the right (so that the string remains taut) to a height $h/2$ from the lowest point and released. If the coefficient of restitution between the balls is $e = m/M$ find the speed of the mass m after the first and second impacts.

Q5 The cue ball in snooker is located at the origin. Both the cue ball and the red ball you are aiming to pot have radius 1. The red ball is centred at $\underline{r} = 2\sqrt{3}\underline{\hat{x}}$, and you strike the cue ball with speed U at an angle $\pi/6$ to the positive x-axis. Find the position of the centre of the cue ball when the two balls collide, and calculate the subsequent velocities of both balls if the collision is perfectly elastic.

Q6 A ping pong ball is held on top of a golf ball and the two are dropped onto the ground from a height of 1m. Assuming the mass of the ping pong ball is negligible, and that all collisions are perfectly elastic, and neglecting air resistance in the subsequent projectile motion, what height will the ping pong ball reach?

MATH1302 Example Sheet 6.

Questions 1, 3 and 5 due in by 11am on Friday 11 March.

Q1 A fisherman, returning from a fishing trip, turns off his engine and lets his dinghy drift slowly at speed U the last few metres to the river bank. When still a few metres from the bank he throws his rucksack mass m towards the bank (in the direction that the boat is drifting) horizontally at speed V relative to the boat. Given that the combined mass of the fisherman, the dinghy and its contents, including the rucksack, is $M+m$, find the condition that he will reach the bank without having to restart the engine. (Assume that the river is stationary.)

Q2 A runaway railfreight carriage filled with sand is travelling freely down a long, straight and constantly inclined railway track. The empty carriage has mass M, it initially holds S_0 mass of sand, and the inclination of the track to the horizontal is α. There is a crack in the carriage which allows sand to leak out at a constant rate. There is also a partially failed brake that provides a constant resistive force B to slow the carriage. Find the speed of the carriage from rest at time $t < T$ where T is the time taken for all the sand to leak out.

Q3 A particle moving along the x-axis under a constant force F gains mass by collecting material that is moving along the x-axis with velocity u. If m and $v = \mathrm{d}x/\mathrm{d}t$ are the mass and the velocity of the particle at time t show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = F + u\frac{\mathrm{d}m}{\mathrm{d}t}.$$

If $u = 0$ and $m = M + kx$ where $M, k > 0$ are constants and the particle starts from rest at the origin, show that $kx = \sqrt{(M^2 + kFt^2)} - M$.

Q4 A raindrop falls vertically through a cloud of water particles which are at rest, and accumulates particles at a rate kv units of mass per unit time when its speed is v. If the raindrop starts from rest and with mass M, show that its speed v after falling through a distance x satisfies

$$v\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{kv^2}{M+kx} = g.$$

Hence find v as a function of x.

Q5 Exam question 2007.

A hailstone falls from rest through a cloud under gravity. Initially it is spherical with radius a. As it falls it accumulates mass at a rate $\pi\rho r^2$, where ρ is its constant density, but remains spherical in shape.

(a) Find the radius of the hailstone at time t.

(b) Show that

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{3v}{t+4a} = g,$$

where g is acceleration due to gravity.

(c) Find $v(t)$.

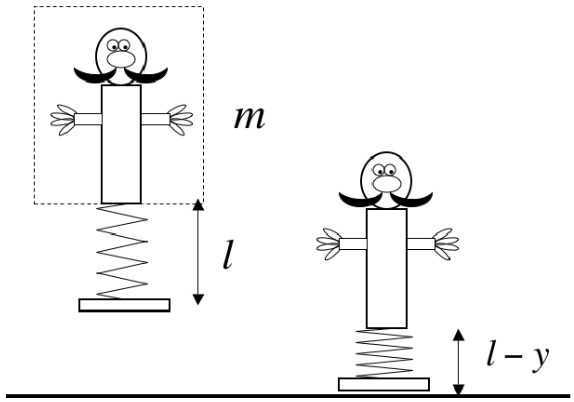
(d) If at $t = t_1$ the hailstone emerges from the cloud into sunshine and continues to fall, but now loses its mass due to evaporation at rate $\pi\rho r^2$, while remaining spherical, find its radius at time $t_2 > t_1$.

MATH1302 Example Sheet 7.

Questions 1, 2 and 4 due in by 11am on Friday 18 March.

Q1 Consider two blocks of mass m and mass $2m$ on a frictionless table connected by a spring with stiffness k and unstretched length ℓ. They are compressed together so that the spring has length $\ell/2$. Find the maximum speeds of the two blocks after they are released from rest.

Q2 Zebedee, the bouncing character from the Magic Roundabout, consists of a body of mass m on top of a light spring of unstretched length ℓ and stiffness k and a light flat circular base.



He is bouncing periodically and vertically on the spot under gravity, and so that the base reaches a maximum height h. Show that the period of his bouncing is

$$2\sqrt{\frac{2h}{g}} + 2\sqrt{\frac{m}{k}}\left\{\pi - \tan^{-1}\sqrt{\frac{2hk}{mg}}\right\}.$$

(Assume that the spring is never fully compressed during the motion, and that the collision of the light plate with the ground is inelastic.)

Q3 A steel ball of mass m is suspended from a fixed beam using an elastic string of natural length ℓ and modulus of elasticity λ. Below the ball is a solenoid which produces a magnetic field that provides a magnetic *pull*

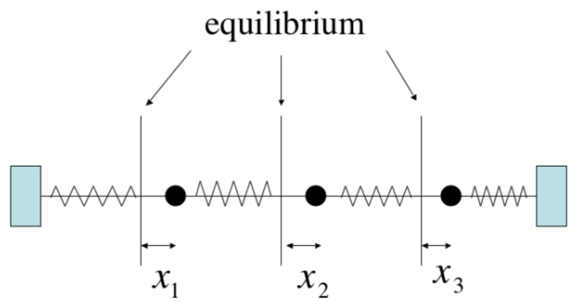
$$F(t) = A(1 + \cos(\omega t)), \qquad (A, \omega > 0)$$

on the steel ball. Initially the ball hangs in equilibrium. The solenoid is then turned on. Show that while the string remains taut the downwards displacement from equilibrium y of the ball satisfies

$$\ddot{y} + \frac{\lambda}{m}y = \frac{A}{m}(1 + \cos(\omega t)),$$

and when $\omega^2 = \lambda/m$ find the ball's displacement while the string remains taut. (Assume that the ball does not collide with the solenoid and that motion is entirely vertical.)

Q4 Consider three masses connected vertically between two plates using four springs, as through 90^o) below.



The springs and masses are identical with stiffness k and mass m respectively. Show that motion of the system about equilibrium is given by

$$\ddot{\underline{x}} = -\underline{\underline{A}}\,\underline{x}$$

where $\underline{x} = (x_1, x_2, x_3)^T$ is the vector of mass displacements from equilibrium, and

$$\underline{\underline{A}} = \begin{pmatrix} 2\sigma & -\sigma & 0 \\ -\sigma & 2\sigma & -\sigma \\ 0 & -\sigma & 2\sigma \end{pmatrix}$$

where $\sigma = k/m$.

Show that the general solution for the mass displacements is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos\left((2\sigma)^{1/2} t + \delta_1\right) + \beta \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos\left((2\sigma - \sqrt{2}\,\sigma)^{1/2} t + \delta_2\right)$$
$$+ \gamma \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos\left((2\sigma + \sqrt{2}\,\sigma)^{1/2} t + \delta_3\right)$$

where α, β, γ and δ_1, δ_2, δ_3 are constants.

Q5 An oscillator operates under the equation

$$\ddot{y} + y = \epsilon \dot{y}^3$$

where the constant $\epsilon \ll 1$.

(a) Verify that the approximation $y = y_1(t) + \epsilon y_2(t)$ gives $\ddot{y}_1 + y_1 = 0$ and $\ddot{y}_2 + y_2 = \dot{y}_1{}^3$.

If $y_1 = \cos t$, show that

(b) the y_2 equation has a first and third harmonic[4] on its right-hand side.

(c) Solve for y_2 given that $y_2(0) = 0$ and $\dot{y}_2(0) = 0$, and deduce that the approximation becomes invalid for large t.

[4]The kth harmonic here is a term of the form $\cos(kt + \delta)$ for some δ.

MATH1302 Example Sheet 8.

Question 6 due in Tuesday 22nd March at the lecture.

Qu 1 *Sketch* the profile of the wave $y(x,t) = e^{-(x-t)^2}$ for $t = 0, 1, 2, 3, 4$. Sketch also the profile $y(x,t) = y_1(x,t) + y_2(x,t)$ of the superposed waves $y_1(x,t) = e^{-(x-t)^2}$, $y_2(x,t) = e^{-(x+t)^2}$, showing the resulting profile for the sequence $t = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

Qu 2 Explain the difference between progressive and standing waves. Show how the standing wave $y(x,t) = A\cos(\alpha x)\sin(\beta t)$ can be produced from two progressive waves. Find the amplitudes, frequencies, wavelengths and speeds of these two progressive waves.

Qu 3 Consider two waves y_1 and y_2 given by $y_1(x,t) = A\cos(k_1(x-c_1t))$ and $y_2(x,t) = A\cos(k_2(x-c_2t))$. Find and simplify an expression for the superposition $y(x,t) = y_1(x,t)+y_2(x,t)$. Describe carefully the progression of the wave $y(x,t)$ when $|k_1 - k_2|$ is small in the cases (i) $c_1 = c_2$ and (ii) $c_1 > c_2$.

Qu 4 For the previous question, find the points (x,t) (for $x \in \mathrm{R}$, $t \geq 0$) where the amplitude of the superposition wave $y(x,t)$ is $2A$.

Qu 5 A harp string of length ℓ is stretched in a straight line between two points. The plucked string vibrates according to the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}, \tag{57}$$

where T is the tension (treated as constant) in the string and $\rho > 0$ is a constant. Show that (57) has standing wave solutions of the form $y(x,t) = A\sin kx\sin(\omega t + \delta)$ and find expressions for k and ω. What does ω represent, and how does it vary as the tension T is increased?

A testing robot can pull the string into any initial profile $y(x,0) = Y(x)$. Verify that for each A_n the function

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right), \tag{58}$$

where $c = \sqrt{T/\rho}$, satisfies both the wave equation (57) and the boundary conditions $y(0,t) = 0$ and $y(\ell,t) = 0$ for $t \geq 0$ and the initial condition that the string is released from rest at $t = 0$.

Prove (by multiplying (58) by $\sin(m\pi x/\ell)$ and integrating) that $A_n = (2/\ell)\int_0^\ell Y(x)\sin(n\pi x/\ell)\,\mathrm{d}x$. What profile $Y(x)$ should the robot pull the string to in order to obtain just the first harmonic?

Qu 6 **2008 Exam Question.**

(a) Explain the difference between standing and progressive waves and describe a real world example of each.

(b) Two waves $y_1(x,t) = A\cos(k_1x - \omega_1t)$ and $y_2(x,t) = A\cos(k_2x - \omega_2t)$ are superposed. Find and simplify an expression for the resulting waveform $y(x,t) = y_1(x,t) + y_2(x,t)$ in terms of ϵ, k, ω, δ, where

$$2\epsilon = k_2 - k_1, \qquad 2k = k_1 + k_2, \qquad 2\delta = \omega_2 - \omega_1, \qquad 2\omega = \omega_1 + \omega_2.$$

(c) Find conditions on ϵ, k, ω, δ such that

(i) y is a standing wave, and

(ii) y is a progressive wave.

(d) Sketch $y(x,0)$ when $\epsilon \ll k$.